Surgery on products. I

by F. J.-B. J. Clauwens

Mathematical Institute, Catholic University, Nijmegen

Communicated by Prof. W. T. van Est at the meeting of June 17, 1978

The aim of this paper is to determine how obstructions for surgery, as introduced by C. T. C. Wall in his celebrated book [W1], behave with respect to the operation of taking the cartesian product with a fixed manifold P. We accomplish this, in contrast to other existing literature, for the case of arbitrary fundamental groups, and in contrast to [R] in a geometrical way.

In the following each manifold and Poincaré-complex will be understood to be equipped with an orientation twisted by the first Stiefel-Whitney class, or equivalently an honest orientation of the orientation covering.

Let a finitely presented group G and a homomorphism $w_G\colon G \to \{\pm 1\}$ be given.

DEFINITION: *An n-manifold over (G, w_G) is a pair consisting of an n-manifold P in the above sense and a homomorphism $\alpha\colon \pi_1(P) \to G$ such that $w_G\alpha$ is the first Stiefel-Whitney class $w_1(P)$ of P.*

There is an obvious concept of cobordism of such objects. We denote the set of equivalence classes by $\Omega_n(G, w_G)$; it is made into an abelian group by the operation of disjoint union.

Equivalently one may look at the covering $\tilde{P}$ of P induced by α together with an orientation and a free action of G on $\tilde{P}$ such that w_G measures the preservation of orientation.

For calculations the following description is the most suitable one. Given a space X with an involution τ we consider the bordism group $\Omega_n^-(X;\tau)$ of n-manifolds together with an honest orientation, an orientation reversing involution T, and an equivariant map to X. Then we take for X the double covering of the classifying space BG which is induced by w_G and for τ the covering transformation. In case w_G is trivial we so recover the classical identification of $\Omega_n(G, 1)$ with $\Omega_n(BG)$.

The following definition is found in chapter 9 of [W1].

DEFINITION: *A surgery problem over (H, w_H) is an object determined by the following data: a finite Poincaré pair $(X, \partial X)$ in the above sense, a compact m-manifold with boundary $(M, \partial M)$ in the same sense, a map $f\colon (M, \partial M) \to (X, \partial X)$ of pairs of degree one, inducing a homotopy equivalence $\partial M \to \partial X$, a bundle ν over X, a stable trivialisation F of $TM \oplus f^*\nu$ and a homomorphism $\beta\colon \pi_1(X) \to H$ such that $w_H\beta$ is $w_1(X)$.*

We will abbreviate this object to f. There is an obvious concept of bordism between such objects; the set of bordism classes is made into an abelian group by disjoint union and is denoted by $L_m(H, w_H)$.

Given objects as above one can construct a new surgery problem over $(G\times H\colon w_G w_H)$ determined by the following data: the finite Poincaré pair $(P\times X, P\times\partial X)$, the $m+n$ manifold with boundary $(P\times M, P\times\partial M)$, the map $id\times f\colon P\times M \to P\times X$, the bundle $\nu P\times\nu$ over $P\times X$, where νP denotes the stable normal bundle of P i.e. a bundle with a framing F' of $TP\oplus\nu P$, the framing $F'\times F$ of $T(P\times M)\oplus(1\times f)^*(\nu P\times\nu) = (TP\oplus \oplus\,\nu P)\times(TM\oplus f^*\nu)$, and the homomorphism $\alpha\times\beta\colon \pi_1(P\times X) = \pi_1(P)\times \times\,\pi_1(X) \to G\times H$.

We will abbreviate this object to $1\times f$. This construction is well defined on classes and induces a biadditive pairing

$$\Omega_n(G, w_G)\times L_m(H, w_H) \to L_{n+m}(G\times H, w_G w_H).$$

In this paper we study this map in the case that n and m are even: $n = 2q$ and $m = 2k$.

We denote by A the integral group ring $Z[G]$ equipped with the antihomomorphic involution – defined by the formula $\overline{\sum n_g g} = \sum w_G(g) n_g g^{-1}$; similarly one constructs B from (H, w_H); then $A\otimes B$ is associated with $(G\times H, w_G w_H)$. We then consider pairs (V, ψ) consisting of a stably free left B module V and a nonsingular $(-1)^k$ symmetric quadratic form ψ on V in the sense of [W2]. There is an obvious notion of direct sum of such pairs. The Grothendieck group quotiented by the subgroup of standard quadratic forms is denoted by $L_{2k}(B)$.

The importance of these concepts stems from the fact that for $k > 2$ a canonical isomorphism $s\colon L_{2k}(H, w_H) \to L_{2k}(B)$ exists. If X is connected and β is an isomorphism (which can be arranged by a bordism of f) then $s(f) = 0$ iff one can change f by surgery (i.e. a bordism that fixes X) into a homotopy equivalence. For this reason $s(f)$ is called the surgery obstruction of f.

In this paper we prove the following

THEOREM: *There exists a free left A module K and a sesquilinear form σ on K such that $\sigma \otimes \psi$ is a nonsingular quadratic form representing $s(1 \times f)$, where ψ is one representing $s(f)$.*

The form σ is nonsingular i.e. $Ad(\sigma)$ is invertible, and is almost $(-1)^q$ symmetric in the sense that $(-1)^q (Ad\,\sigma)^{-1}(Ad\,\sigma)\dagger - 1$ is nilpotent. Furthermore (K, σ) can be expressed in terms of the Miscenko/Ranicki [R] *symmetric Poincaré complex associated with P.*

This tensor product of forms must be understood in a graded sense i.e. $(\sigma \otimes \psi)(a \otimes x, b \otimes y) = (-1)^{kq}\, \sigma(a, b) \otimes \psi(x, y)$.

The paper is organized as follows: in section 1 we introduce some notations; in section 2 we perform the low-dimensional surgery on $id \times f$; in section 3 we prove a few technical lemmas which are needed for the computation of the mid-dimensional homology of the resulting surgery-problem in section 4. In section 5 we establish a relation between the data we used about P and the description of P on the chain level as in [R]; in section 6 we describe the form σ in these terms. In section 7 we show how to represent homology-classes by immersions in the right regular homotopy class and in section 8 we count the intersections, thereby finishing the proof of the main theorem. Finally in section 9 we give another description of the form σ and an example of application of the theorem.

§ 1. DEFINITIONS AND NOTATIONS

We now consider a surgery problem f over (H, w_H) as in § 0. By a bordism of f we can arrange X to be connected and β to be an isomorphism (see [W1], p. 91), hence $\tilde{X}$ is connected and simply connected. The groups $H_i(\tilde{M})$ and $H_i(\tilde{X})$ have the structure of left $B = Z[H]$ modules because of the left H actions on $\tilde{M}$ and $\tilde{X}$; $f_* : H_i(\tilde{M}) \to H_i(\tilde{X})$ is a module homomorphism and is surjective since f is of degree one. We denote the kernel by $K_i(M)$.

By doing preliminary surgery we may suppose that f is k-connected, hence that $\tilde{M}$ is connected and simply connected and that $K_i(M)$ vanishes unless $i = k$; furthermore we may assume that $K_k(M)$ is a free B module with basis $e_1, \ldots, e_r$ say ([W1], p. 49).

Each $e_j \in K_k(M)$ can be represented by an immersion $g_j : S^k \times D^k(1) \to \tilde{M}$ together with a nullhomotopy h_j of $f \circ g_j(\ , 0)$. The regular homotopy class of g_j is well determined by the condition that the stable framing of $g_j(\ , 0)^* TM$ induced by the derivative of g_j, together with the restriction to S^k of the canonical framing of $h_j^* \nu$ corresponds under $g_j(\ , 0)^*$ with the given framing F of $TM \oplus f^* \nu$.

By general position we may assume that the g_j intersect regularly i.e. on an appropriate coordinate chart the intersection looks like $(\mathbf{R}^k \times 0, 0 \times \mathbf{R}^k)$ in $\mathbf{R}^{2k}$. By choosing a Riemann structure on M which is Euclidean on the above-mentioned chart and by using the exponential map to redefine g_j we may assume that the g_j are disjoint embeddings except that for certain pairs of points (p, p') and coordinate maps $\eta_p: (D^k(1), 0) \to (S^k, p)$ and $\eta_{p'}$ around those points and for certain $\gamma \in H$ we have:

$$g_j(\eta_p(x), y) = \gamma^{-1} g_{j'}(\eta_{p'}(y), x) \text{ for all } x, y \text{ in } D^k(1).$$

We write D_p and $D_{p'}$ for the images of η_p and $\eta_{p'}$; $D_p(R)$ denotes the η_p image of the disc of radius R.

Now we choose a C^∞ function $\varkappa$ with values in $[0, 1]$ on each copy S^k_j of the standard k-sphere such that for any intersection-pair $\{p, p'\}$ as above $\varkappa = 0$ on $D_p(\frac{1}{2})$ and $\varkappa = 1$ on $D_{p'}(\frac{1}{2})$ or vice versa, and such that $\varkappa$ vanishes outside the images of the η.

Now we define ψ by the formula $\psi(e_j, e_{j'}) = \sum_{\gamma \in H} (g_j \cdot \gamma^{-1} g_{j'})_< \gamma$, where $\cdot$ denotes the ordinary intersection-number, which counts the number of pairs (p, p') as above with multiplicity $+1$ or -1 depending on whether $\eta_{p'}$ preserves or changes orientation, and $<$ means that we count only those pairs for which $\varkappa(p) < \varkappa(p')$, hence $\varkappa(p) = 0$ and $\varkappa(p') = 1$.

If we count all pairs, we get the equivariant intersection-number $\lambda(e_j, e_{j'})$. Since η_p changes orientation by a factor $\varepsilon_p = (-1)^k w_H(\gamma)\ \varepsilon_{p'}$ if $\eta_{p'}$ does so by a factor $\varepsilon_{p'}$, we see that $\lambda(e_j, e_{j'}) - \psi(e_j, e_{j'}) = (-1)^k\, \overline{\psi(e_{j'}, e_j)}$: a pair $\{p, p'\}$ with $\varkappa(p) > \varkappa(p')$ such that $g_j(\eta_p(x), y) = \gamma^{-1} g_{j'}(\eta_{p'}(y), x)$ contributes $\varepsilon_{p'}\gamma$ to $\lambda(e_j, e_{j'})$; it can also be seen as a pair with $\varkappa(p') < \varkappa(p)$ such that $g_{j'}(\eta_{p'}(y), x) = \gamma g_j(\eta_p(x), y)$ and so it contributes $\varepsilon_p \gamma^{-1}$ to $\psi(e_{j'}, e_j)$ hence $\varepsilon_{p'}\gamma = (-1)^k\, \varepsilon_p \bar{\gamma}^{-1}$ to $(-1)^k\, \overline{\psi(e_{j'}, e_j)}$.

This ψ extends to a pairing $K_k(M) \times K_k(M) \to B$ which is sesquilinear, i.e. biadditive and such that $\psi(ax, by) = b\psi(x, y)\bar{a}$ for $x, y \in K_k(M)$ and $a, b \in B$. For any left B module V, the dual $V^d = \mathrm{Hom}_B(V, B)$ has the structure of a left B module such that $(af)(v) = f(v)\bar{a}$ for $a \in B$, $f \in V^d$, $v \in V$; in particular this applies to $V = K_k(M)$. Saying that ψ is sesquilinear is equivalent to saying that the map $Ad(\psi): V \to V$ defined by the formula $((Ad\psi)x)(y) = \psi(x, y)$ is a module homomorphism.

The same applies to the symmetrisation λ of ψ; since $Ad(\lambda)$ is an isomorphism one calls ψ a nonsingular quadratic form [W2]. The class of ψ in $L_{2k}(B)$ is independent of choices and defines $s(f)$ (see [W1], p. 50).

For later use we introduce the notation $\tilde{\psi}$ for the homomorphism $V \to V$ corresponding to Ad under the isomorphism $V \to V^d$ mapping the free generators e_j of V to their duals e_j^*; thus $\tilde{\psi}(x) = \sum_{j=1}^r \overline{\psi(x, e_j)} e_j$. An intersection-pair $\{p, p'\}$ with $\varkappa(p) < \varkappa(p')$ as above contributes $(-1)^k \varepsilon_p \gamma^{-1} e_{j'}$ to $\tilde{\psi}(e_j)$. Notice that $\lambda(\tilde{\lambda}^{-1}\tilde{\psi}x, y) = \psi(x, y)$.

Now consider P. We suppose P triangulated; we denote the i-skeleton by P_i. In any C^∞ neighbourhood of the identity one can find a diffeomorphism $\xi: P \to P$ which puts each simplex of P in transverse position with respect to the simplices of P. In particular $\xi P_t \cap P_{n-t-1} = \emptyset$, $\xi P_t \cap P_{n-t}$ is discrete.

If we choose ξ close enough to the identity, then we can find an isotopy ξ_t such that $\xi_0 = id$ and $\xi_1 = \xi$: we embed P in some Euclidean space, connect x and ξx by a straight line segment, and project down to P; if ξ was chosen close enough to id this yields diffeomorphisms.

Now consider the path in the space of C^∞ maps $P \to P$ which is defined by ξ_{2t-1} for $t \in [\frac{1}{2}, 1]$ and by ξ_{1-2t}^{-1} for $t \in [0, \frac{1}{2}]$: in any neighbourhood of it we can find a path χ with the same endpoints such that χ puts the product of $[0, 1]$ and any simplex of P in transverse position with respect to the simplices of P, and homotopic with it. If the path is sufficiently close it consists entirely of diffeomorphisms.

Furthermore one can choose regular neighbourhoods S of P_{q-2}, Q of P_{q-1} and R of P_q small enough so that

$$\begin{aligned}
&\chi([0, 1] \times R) \cap S = \chi([0, 1] \times Q) \cap Q = \emptyset \\
&\xi R \cap Q = \xi Q \cap R = \emptyset \\
&S \subset \operatorname{int}(Q) \text{ and } Q \subset \operatorname{int}(R).
\end{aligned}$$

We denote by $\tilde{P}_t$ the covering of P_t induced by $\tilde{P}$; idem for Q, S etc. Notice that our diffeomorphisms, being nullhomotopic, define unique diffeomorphisms of $\tilde{P}$ with similar properties.

§ 2. SURGERY BELOW THE MIDDLE DIMENSION

We define a map $\tilde{\Omega}_j: \tilde{P} \times S^k \times D^k(\frac{1}{2}) \to \tilde{P} \times \tilde{M}$ by the formula

$$\tilde{\Omega}_j(y, x, v) = (\xi_{\varkappa(x)} y, g_j(x, v)).$$

The $\tilde{\Omega}_j$ determine disjoint embeddings $\Omega_j: Q \times S^k \times D^k(\frac{1}{2}) \to P \times M$. For suppose that $\Omega_j(y, x, v) = (\theta \times \gamma)^{-1}\, \Omega_{j'}(y', x', v')$: then $\xi_{\varkappa(x)} y = \theta^{-1} \xi_{\varkappa(x')} y'$ and $g_j(x, v) = \gamma^{-1} g_{j'}(x', v')$. Unless $x = x'$, $v = v'$ and $y = \theta^{-1} y'$ the last formula implies that for some intersection-pair $\{p, p'\}$ we have $x \in D_p$, $x' \in D_{p'}$, or vice versa i.e. $\varkappa(x) = 0$, $\varkappa(x') = 1$ or vice versa; hence we get a contradiction with $\xi Q \cap Q = \emptyset$.

Hence we can use Ω to define a manifold by glueing:

$$W = P \times M \times [0, 1] \cup \bigcup_{j=1}^{r} Q \times D_j^{k+1} \times D^k(\tfrac{1}{2}).$$

Since Ω_j is homotopic to $1 \times g_j$ we can extend $1 \times f: P \times M \to P \times X$ to a map $W \to P \times X$; similarly we can extend the framing F, and $\alpha \times \beta$ extends to an isomorphism $\pi_1(W, *) \cong G \times H$ by the van Kampen theorem.

Now ∂W is the disjoint union of $\partial_- W \cong P \times M$ and $N = \partial_+ W \cong$ $\cong (P \times M - \operatorname{im} \Omega) \cup \bigcup_{j=1}^{r} (Q \times D_j^{k+1} \times S^{k-1}(\frac{1}{2}) \cup Q \times D_j^{k+1} \times D^k(\frac{1}{2}))$.

THEOREM 1: *The surgery problem $N \to P \times X$ thus obtained is $(q+k)$-connected.*

PROOF: From now on we write $K_i(W)$ for the kernel of the above-mentioned map in homology: $H_i(\tilde{W}) \to H_i(\tilde{P} \times \tilde{X})$; similarly one has $K_i(P \times M)$, $K_i(N)$ etc.

We can construct a module map $H_i(\tilde{P}) \otimes K_k(M) \to K_{i+k}(P \times M)$ which maps $\{c\} \otimes e_j$ to the class represented by $c \times g_j(S^k \times 0)$; this is an isomorphism by the Kunneth theorem.

Similarly we have a map

$$H_i(\tilde{Q}) \otimes K_k(M) \to K_{i+k+1}(W, P \times M \times [0, 1]) \cong K_{i+k+1}(W, P \times M)$$

mapping $\{c\} \otimes e_j$ to the class represented by $c \times D_j^{k+1} \times 0$. This map is isomorphic by excision and the Kunneth theorem.

From the way these maps are defined it follows that the following diagram commutes

$$\begin{array}{ccc} H_i(\tilde{Q}) \otimes K_k(M) & \longrightarrow & H_i(\tilde{P}) \otimes K_k(M) \\ \downarrow & & \downarrow \\ K_{k+i+1}(W, P \times M) & \longrightarrow & K_{k+i}(P \times M) \end{array}$$

where the upper horizontal arrow is induced by the inclusion $Q \subset P$. Since $H_i(\tilde{P}, \tilde{Q}) = H_i(\tilde{P}, \tilde{P}_{q-1}) = 0$ for $i \leqslant q-1$, that map is a surjection for $i \leqslant q-1$ and an injection for $i < q-1$, hence ∂ is, as well. So $K_{k+i}(W)$ vanishes for $k+i \leqslant k+q-1$.

It is clear that the union of N and $\bigcup_{j=1}^{r} Q \times D_j^{k+1} \times D^k(\frac{1}{2})$ is a retract of W relative to N; the intersection of these two is

$$\bigcup_{j=1}^{r} \{Q \times D_j^{k+1} \times S^{k-1}(\tfrac{1}{2}) \cup \partial Q \times D_j^{k+1} \times D^k(\tfrac{1}{2})\}.$$

Accordingly, we see that by retraction, excision and the Kunneth theorem we have an isomorphism $H_i(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \to K_{i+k}(W, N)$ mapping $\{c\} \otimes e_j$ to the class of $c \times 0_j \times D^k$.

Since $H_i(\tilde{Q}, \partial\tilde{Q}) \cong H_c^{n-i}(\tilde{Q}) \cong H_c^{n-i}(\tilde{P}_{q-1})$ vanishes for $n-i \geqslant q$ i.e. $i \leqslant q$, it follows that $K_{i+k}(W, N) = 0$ for $i+k \leqslant q+k$.

Substituting the above results in the long exact homology sequence of the pair (W, N) we deduce that $K_i(N) = 0$ for $i \leqslant q+k-1$. Furthermore the inclusion of N in W induces an isomorphism of fundamental groups: the inclusion $\partial Q \times D^{k+1} \times S^{k-1} \subset \partial Q \times D^{k+1} \times D^k$ does, hence $Q \times D^{k+1} \times S^{k-1} \subset Q \times D^{k+1} \times S^{k-1} \cup \partial Q \times D^{k+1} \times D^k$ does by van Kampen, hence $Q \times D^{k+1} \times S^{k-1} \cup \partial Q \times D^{k+1} \times D^k \subset Q \times D^{k+1} \times D^k$ does, hence $N \subset W$ does.

For a similar reason the inclusion $P \times M \to W$ induces an isomorphism of fundamental groups. Hence the isomorphism of $\pi_1(M)$ and $\pi_1(X)$ implies one of $\pi_1(N)$ and $\pi_1(P \times X)$. Q.E.D.

§ 3. SOME MAPS AND DIAGRAMS

In this section we prove some results which are needed for the determination of $K_{q+k}(N)$ in the next section.

LEMMA 1: *$K_{t+k}(W)$ is isomorphic to $H_t(\tilde{P}, \tilde{Q}) \otimes K_k(M)$.*

PROOF: First we note the existence of a homomorphism

$$H_t(\tilde{P}, \tilde{Q}) \otimes K_k(M) \to K_{t+k}(W),$$

mapping $\{c\} \otimes e_j$ to $\tilde{\Omega}_j(c \times S^k \times 0) \cup \partial c \times D_j^{k+1} \times 0 \subset \tilde{W}$.

Then the following diagram commutes by construction:

$$\begin{array}{ccccccc}
\dots H_{t+1}(\tilde{P}, \tilde{Q}) \otimes K_k(M) & \to & H_t(\tilde{Q}) \otimes K_k(M) & \to & H_t(\tilde{P}) \otimes K_k(M) & \to & H_t(\tilde{P}, \tilde{Q}) \otimes K_k(M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{t+k+1}(W) & \to & K_{t+k+1}(W, P \times M) & \to & K_{t+k}(P \times M) & \to & K_{t+k}(W)
\end{array}$$

It follows that the map is an isomorphism, since we have seen in the proof of theorem 1 that the other vertical maps are. Q.E.D.

LEMMA 2: *There is a commutative ladder*

$$\begin{array}{ccccccc}
H_t(\tilde{P} - \tilde{Q}) \otimes K_k(M) & \to & H_t(\tilde{P}) \otimes K_k(M) & \to & H_t(\tilde{P}, \tilde{P} - \tilde{Q}) \otimes K_k(M) & \to & H_{t-1}(\tilde{P} - \tilde{Q}) \otimes K_k(M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{t+k}(N) & \longrightarrow & K_{t+k}(W) & \longrightarrow & K_{t+k}(W, N) & \longrightarrow & K_{t+k-1}(N)
\end{array}$$

Here the horizontal maps are the usual ones.

PROOF: We define the vertical arrows in step (a) and discuss the commutativity of the three squares in the remaining three steps.

Step (a) We define a map $\tilde{\Psi}_j \colon \tilde{P} \times S^k \times D^k \to \tilde{P} \times \tilde{M}$ analoguous to $\tilde{\Omega}_j$ by the formula $\tilde{\Psi}_j(y, x, v) = (\xi_{1-\varkappa(x)} y, g_j(x, v))$. If $w \in D^k$ is of length $\frac{1}{2}$ then $\Psi_j((P - Q) \times S^k \times w) \subset P \times M$ is contained in the closure of $P \times M - \operatorname{im} \Omega$, hence in N. For suppose that $\tilde{\Psi}_j(y, x, w) = (\theta \times \gamma)^{-1} \tilde{\Omega}_{j'}(y', x', v')$, then $\xi_{1-\varkappa(x)} y = \theta^{-1} \xi_{\varkappa(x')} y'$ and $g_j(x, w) = \gamma^{-1} g_j(x', v')$. The last formula implies that either $j = j'$, $x = x'$, $w = v'$, in contradiction with $v' \in \operatorname{int}(D^k(\frac{1}{2}))$, or for some intersection-pair $\{p, p'\}$ we have $x \in D_p(\frac{1}{2})$, $x' \in D_{p'}(\frac{1}{2})$ or vice versa i.e. $\varkappa(x) = 0$, $\varkappa(x') = 1$ or vice versa. But then the first formula says that $y = \theta^{-1} y'$, contradicting the fact that $y \in \tilde{P} - \tilde{Q}$ and $y' \in \tilde{Q}$. Thus Ψ determines a homomorphism $H_t(\tilde{P} - \tilde{Q}) \otimes K_k(M) \to K_{t+k}(N)$.

The second vertical arrow is the composition of the Kunneth isomorphism $H_t(\tilde{P}) \otimes K_k(M) \cong K_{t+k}(P \times M)$ and the inclusion $\tilde{P} \times \tilde{M} \to \tilde{W}$. The third vertical map is the composition of the isomorphism $H_t(\tilde{Q}, \partial \tilde{Q}) \otimes K_k(M) \to$ $\to K_{t+k}(W, N)$ discussed in the proof of theorem 1, and the tensor-product of the excision isomorphism $H_t(\tilde{Q}, \partial \tilde{Q}) \cong H_t(\tilde{P}, \tilde{P} - \tilde{Q})$ with $(-1)^k \tilde{\lambda}$.

Step (b) The commutativity of the first square is an immediate consequence of the fact that $\tilde{\Psi}_j$ is isotopic to $1 \times g_j$ in $\tilde{P} \times \tilde{M} \subset \tilde{W}$.

Step (*c*) We have to prove the commutativity of the following diagram:

$$\begin{tikzcd}
H_t(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \arrow[r] \arrow[d, "(-1)^k\tilde{\lambda}"] & H_t(\tilde{P}, \tilde{P}-\tilde{Q}) \otimes K_k(M) \arrow[r, "\partial"'] & H_{t-1}(\tilde{P}-\tilde{Q}) \otimes K_k(M) \arrow[ddd] \\
H_t(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \arrow[dd] & & \\
 & & \\
K_{t+k}(W, N) \arrow[rr, "\partial"] & & K_{t+k-1}(N)
\end{tikzcd}$$

If we start with $\{c\} \otimes e_j \in H_t(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M)$ and go along the upper side we get the class of $\tilde{\Psi}_j(\partial c \times S^k \times w) = \partial\tilde{\Psi}_j(c \times S^k \times w)$ in $K_{t+k-1}(N)$.

Let Δ denote the union of the $D_p(\frac{1}{2}) \subset S^k$, where p runs through the intersection points of g_j with the $g_{j'}$. We will see that $\tilde{\Psi}_j(c \times \Delta \times w)$ is precisely the part of $\tilde{\Psi}_j(c \times S^k \times w) \subset \tilde{P} \times \tilde{M}$ which lies in im $(\tilde{\Omega})$; then $\tilde{\Psi}_j(c \times (S^k - \Delta) \times w)$ lies in $\tilde{N}$ and can thus be viewed as an homology between $\partial\tilde{\Psi}_j(c \times \Delta \times w)$ and $\partial\tilde{\Psi}_j(c \times S^k \times w)$; subsequently we rewrite $\tilde{\Psi}_j(c \times \Delta \times w)$ in terms of $\tilde{\Omega}$ and note that the result corresponds to the other composition in the above diagram.

To prove these assertions we note that $\tilde{\Psi}_j(y, x, w) = (\theta \times \gamma)^{-1}\tilde{\Omega}_{j'}(y', x', v')$ implies that $g_j(x, w) = \gamma^{-1}g_{j'}(x', v')$, hence that $x \in D_p(\frac{1}{2})$, $x' \in D_{p'}(\frac{1}{2})$ for some intersection-pair $\{p, p'\}$; in particular, $x \in \Delta$.

On the other hand such a pair with $g_j(\eta_p(a), b) = \gamma^{-1}g_{j'}(\eta_p(b), a)$ for $a, b \in D^k(\frac{1}{2})$ gives a contribution $\varepsilon_{p'}\gamma = (-1)^k w(\gamma)\varepsilon_p\gamma$ to $\lambda(e_j, e_{j'})$, hence $\varepsilon_p\gamma^{-1}e_{j'}$ to $(-1)^k\tilde{\lambda}(e_j)$. Also we can identify the part $\varepsilon_p\tilde{\Psi}_j(c \times D_p(\frac{1}{2}) \times w)$ of $\tilde{\Psi}_j(c \times S^k \times w)$ with $\varepsilon_p\gamma^{-1}\tilde{\Omega}_{j'}(c \times p' \times D^k(\frac{1}{2}))$ since $\varkappa(p') = 1 - \varkappa(p)$; this represents $\varepsilon_p\gamma^{-1}e_{j'}$ in $K_{t+k}(W, N)$.

Step (*d*) We can use the foregoing calculation to prove the commutativity of the following diagram

$$\begin{tikzcd}
H_t(\tilde{P}) \otimes K_k(M) \arrow[r] \arrow[ddd] & H_t(\tilde{P}, \tilde{P}-\tilde{Q}) \otimes K_k(M) & H_t(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \arrow[l] \arrow[d, "(-1)^k\tilde{\lambda}"] \\
 & & H_t(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \arrow[dd] \\
 & & \\
K_{t+k}(W) \arrow[rr] & & K_{t+k}(W, N)
\end{tikzcd}$$

An element $\{c\}$ of $H_t(\tilde{P})$ can be represented by $d_1 + d_2$ where $d_1 \subset \tilde{Q}$ and $d_2 \subset$ closure of $\tilde{P} - \tilde{Q}$. The result on $\{c\} \otimes e_j$ of going along the left and lower side to $K_{t+k}(W, N)$ is the image of $c \times S^k \times 0$ under $1 \times g_j$ or equivalently under $\tilde{\Psi}_j$. All but $\tilde{\Psi}_j(d_1 \times \Delta \times w)$ lands in $\tilde{N}$ and can be neglected; on the other hand, $\varepsilon_p\tilde{\Psi}_j(d_1 \times D_p(\frac{1}{2}) \times w)$ can be replaced by $\varepsilon_p\gamma^{-1}\tilde{\Omega}_{j'}(d_1 \times \times p' \times D^k(\frac{1}{2}))$ and those terms represent $(-1)^k\tilde{\lambda}(e_j)$ as we have seen above.

Q.E.D.

LEMMA 3: *There is a map $H_i(\tilde{R}, \tilde{Q}) \otimes K_k(M) \to K_{i+k}(N)$ such that the following diagrams commute:*

$$\begin{tikzcd}
H_i(\tilde{R},\tilde{Q})\otimes K_k(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "(a)"] & H_i(\tilde{P},\tilde{Q})\otimes K_k(M) \arrow[d] \\
K_{i+k}(N) \arrow[r] & K_{i+k}(W)
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
H_i(\tilde{R})\otimes K_k(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & H_i(\tilde{R},\tilde{Q})\otimes K_k(M) \arrow[d] \\
H_i(\tilde{P}-\tilde{Q})\otimes K_k(M) \arrow[r] & K_{i+k}(N)
\end{tikzcd}$$

where the map $H_i(R) \to H_i(P-\tilde{Q})$ is induced by ξ.

PROOF: We start by defining the homomorphism $\omega: H_i(R, \tilde{Q}) \otimes \otimes K_k(M) \to K_{i+k}(N)$ which maps $\{c\} \otimes e_j$ to the class of $\tilde{\Omega}_j(c \times S^k \times 0) \cup \cup\, \partial c \times D_j^{k+1} \times 0$ provided we choose the representing cycle c in the closure of $\tilde{R}-\tilde{Q}$; that we end up in $\tilde{N}$ follows from the fact that $\xi R \cap Q = R \cap \cap\, \xi Q = \emptyset$. By construction, then, the diagram (a) commutes. Now we consider the two compositions U and V in diagram (b). There is a homomorphism $H_i(\tilde{R}) \to H_{i+1}(\tilde{P}, \tilde{P}-\tilde{Q})$ which is induced by the map

$$([0, 1] \times \tilde{R}, \{0, 1\} \times R) \to (\tilde{P}, \tilde{P}-\tilde{Q})$$

which maps (t, y) to $\xi^{-1}\xi_t\xi_t y$; furthermore $H_{i+1}(\tilde{P}, \tilde{P}-\tilde{Q}) \cong H_{i+1}(\tilde{Q}, \partial\tilde{Q})$ by excision. Together with $(-1)^k\tilde{\psi}: K_k(M) \to K_k(M)$ and the identification $H_{i+1}(\tilde{Q}, \partial\tilde{Q}) \otimes K_k(M) \cong K_{i+k+1}(W, N)$, this defines a map

$$T: H_i(\tilde{R}) \otimes K_k(M) \to K_{i+k+1}(W, N).$$

We assert that $U - V = \partial T$; this is illustrated in the following diagram:

$$\begin{tikzcd}
H_i(\tilde{R})\otimes K_k(M) \arrow[d] \arrow[ddr, "U"'] \arrow[ddrr, "V"] \arrow[dddd, bend right=60, "T"'] & & \\
H_{i+1}(\tilde{P}, \tilde{P}-\tilde{Q})\otimes K_k(M) \arrow[dd, leftrightarrow] & & \\
 & H_i(\tilde{P}-\tilde{Q})\otimes K_k(M) \arrow[dd] & H_i(\tilde{R},\tilde{Q})\otimes K_k(M) \arrow[ddl] \\
H_{i+1}(\tilde{Q}, \partial\tilde{Q})\otimes K_k(M) \arrow[d] & & \\
K_{i+k+1}(W, N) \arrow[r, "\partial"'] & K_{i+k}(N) &
\end{tikzcd}$$

This assertion will be proved in the same style as in the foregoing. We define a map $\tilde{\Gamma}_j: [0, 1] \times \tilde{P} \times S^k \times D^k(\tfrac{1}{2}) \to \tilde{P} \times \tilde{M}$ by the formula

$$\tilde{\Gamma}_j(t, y, x, v) = (\xi_{(1-\varkappa(x))t}\xi_{t+(1-t)\varkappa(x)}y,\; g_j(x, v)),$$

then

$$\tilde{\Gamma}_j(0, y, x, v) = \tilde{\Omega}_j(y, x, v) \text{ and } \tilde{\Gamma}_j(1, y, x, v) = \tilde{\Psi}_j(\xi y, x, v).$$

In this situation, Δ denotes the union of the $D_p(\tfrac{1}{2}) \subset S^k$ where p belongs to an intersection-pair $\{p, p'\}$ such that $\varkappa(p) < \varkappa(p')$ i.e. $\varkappa(p) = 0$. Then $\tilde{\Gamma}_j([0,1] \times c \times (S^k - \Delta) \times w)$ is a homology in $\tilde{N}$ between

$$\partial \tilde{\Gamma}_j([0,1] \times c \times \Delta \times w) \subset \text{closure im } (\tilde{\Omega})$$

and

$$\begin{aligned}\partial \tilde{\Gamma}_j([0,1] \times c \times S^k \times w) &= \tilde{\Gamma}_j(1 \times c \times S^k \times w) - \tilde{\Gamma}_j(0 \times c \times S^k \times w) = \\ &= \tilde{\Psi}_j(\xi c \times S^k \times w) - \tilde{\Omega}_j(c \times S^k \times w),\end{aligned}$$

which represents $(U - V)(c \otimes e_j)$.

To prove these claims we note that $\tilde{\Gamma}_j(t, y, x, w) = (\theta \times \gamma)^{-1} \tilde{\Omega}_{j'}(y', x', v')$ implies $g_j(x, w) = \gamma^{-1} g_{j'}(x', v')$ hence $x \in D_p(\frac{1}{2})$, $x' \in D_{p'}(\frac{1}{2})$ for some intersection-pair $\{p, p'\}$. In case $\varkappa(x) = 1$, $\varkappa(x') = 0$ we further get $\xi_1 y = \theta^{-1} y'$ contradicting $\xi y \in \xi \tilde{R}$, $y' \in \tilde{Q}$, $\xi \tilde{R} \cap \tilde{Q} = \emptyset$; hence $\varkappa(x) = 0$ i.e. $x \in \Delta$.

On the other hand such a pair, with $g_j(\eta_p(a), b) = \gamma^{-1} g_{j'}(\eta_{p'}(b), a)$ for $a, b \in D^k(\frac{1}{2})$ gives a contribution $\varepsilon_{p'}\gamma$ to $\psi(e_j, e_{j'})$ hence $\varepsilon_p \gamma^{-1} e_{j'}$ to $(-1)^k \tilde{\psi}(e_j)$. Also we can identify $\varepsilon_p \tilde{\Gamma}_j(t \times c \times D_p(\frac{1}{2}) \times w)$ with $\varepsilon_p \gamma^{-1} \tilde{\Omega}_{j'}(\xi^{-1} \xi_t \xi_t c \times 0 \times \times D^k(\frac{1}{2}))$; hence $\tilde{\Gamma}_j([0,1] \times c \times \Delta \times w)$ represents $T(c \otimes e_j)$.

The map $\Xi \colon [0,1] \times [0,1] \times \tilde{P} \to \tilde{P}$ defined by the formula

$$\begin{aligned}\Xi(s, t, y) &= \xi_{1-2st}^{-1}\, \xi_{t(1-s)}\, \xi_{t(1-s)} y \quad \text{for } t \leq \tfrac{1}{2} \\ &\ \xi_{1-s}^{-1}\, \xi_{t(1-s)}\, \xi_{t+s(t-1)} y \quad \text{for } t \geq \tfrac{1}{2}\end{aligned}$$

satisfies

$$\begin{aligned}&\Xi(0, t, y) = \xi^{-1} \xi_t \xi_t y \quad \text{for all } t, \\ &\Xi(s, 0, y) = \xi^{-1} y \ \text{ and } \ \Xi(s, 1, y) = \xi y, \quad \text{for all } s.\end{aligned}$$

Accordingly, we may replace the map $(t, y) \to \xi^{-1} \xi_t \xi_t y$ in the above statement by the map $(t, y) \to \xi_{2t-1}$ for $t \geq \frac{1}{2}$ and ξ_{1-2t}^{-1} for $t \leq \frac{1}{2}$, hence by $(t, y) \to \chi_t y$.

This has the advantage that it shows that T factorizes over $H_t(\tilde{R}, \tilde{Q})$. The above statement can now be read to say that $U = V$ provided we correct the original map $H_t(\tilde{R}, \tilde{Q}) \otimes K_k(M) \to K_{t+k}(N)$ by the term given by either composition in the diagram

$$\begin{tikzcd}
H_t(\tilde{R}, \tilde{Q}) \otimes K_k(M) \arrow[d, "\chi \otimes (-1)^k \tilde{\psi}"] & & \\
H_{t+1}(\tilde{P}, \tilde{P} - \tilde{Q}) \otimes K_k(M) \arrow[rr, "\partial \otimes (-1)^k \tilde{\lambda}^{-1}"'] & & H_t(P - Q) \otimes K_k(M) \arrow[d] \\
H_{t+1}(\tilde{Q}, \partial \tilde{Q}) \otimes K_k(M) \arrow[u, "\cong"'] \arrow[r] & K_{t+k+1}(W, N) \arrow[r] & K_{t+k}(N)
\end{tikzcd}$$

Notice that $\tilde{\lambda}^{-1} \tilde{\psi} = (Ad\, \lambda)^{-1} (Ad\, \psi)$ is independent of the choice of base $\{e_j\}$.

This concludes the proof of the commutativity of (b); the commutativity of (a) is not disturbed by the addition of the correction term to ω.

Q.E.D.

§ 4. THE COMPUTATION OF $K_{q+k}(N)$

We define K to be the cokernel of the map

$$(-\xi, 1): H_q(\tilde{R}) \to H_q(\tilde{P}-\tilde{Q}) \oplus H_q(\tilde{R}, \tilde{Q}).$$

THEOREM 2: *$K_{q+k}(N)$ is isomorphic to $K \otimes K_k(M)$.*

PROOF: According to Lemma 3b there are maps

$$U: H_q(\tilde{P}-\tilde{Q}) \otimes K_k(M) \to K_{q+k}(N)$$

and

$$V: H_q(\tilde{R}, \tilde{Q}) \otimes K_k(M) \to K_{q+k}(N)$$

which agree on $H_q(\tilde{R}) \otimes K_k(M)$; hence there is an induced map

$$K \otimes K_k(M) \to K_{q+k}(N).$$

This will be shown to be an isomorphism.

The following diagram has exact rows

$$\begin{array}{ccccccc}
H_{q+1}(\tilde{P}, \tilde{R}) & \xrightarrow[\partial]{} & H_q(\tilde{R}) & \longrightarrow & H_q(\tilde{P}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \\
H_{q+1}(\tilde{P}, \tilde{R}) & \xrightarrow{\partial} & H_q(\tilde{R}, \tilde{Q}) & \longrightarrow & H_q(\tilde{P}, \tilde{Q}) & \longrightarrow & 0
\end{array}$$

and the vertical maps are injective so

$$H_q(\tilde{R}, \tilde{Q})/H_q(\tilde{R}) \cong H_q(\tilde{P}, \tilde{Q})/H_q(\tilde{P}).$$

The upper row of the next diagram is exact

$$\begin{array}{ccccc}
H_{q+1}(\tilde{P}, \tilde{P}-\tilde{Q}) & \xrightarrow[\partial]{} & H_q(\tilde{P}-\tilde{Q}) & \longrightarrow & H_q(\tilde{P}) \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\
H_{q+1}(\tilde{P}, \tilde{P}-\tilde{Q}) & \xrightarrow{(\partial, 0)} & K = \text{coker}\,(-\xi, 1) & \longrightarrow & H_q(\tilde{P}, \tilde{Q})
\end{array}$$

The vertical maps are injective and their cokernels are isomorphic as we have just seen; hence the lower row is exact. So we get an exact sequence

$$0 \to H_{q+1}(\tilde{P}) \to H_{q+1}(\tilde{P}, \tilde{P}-\tilde{Q}) \to K \to H_q(\tilde{P}, \tilde{Q}) \to 0$$

Now consider the diagram

$$\begin{array}{ccccccccc}
H_{q+1}(\tilde{P}) \otimes K_k(M) & \to & H_{q+1}(\tilde{P}, \tilde{P}-\tilde{Q}) \otimes K_k(M) & \to & K \otimes K_k(M) & \to & H_q(\tilde{P}, \tilde{Q}) \otimes K_k(M) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K_{q+k+1}(W) & \to & K_{q+k+1}(W, N) & \to & K_{q+k}(N) & \to & K_{q+k}(W) & \to & 0
\end{array}$$

Of this diagram we know the following:

i) as we have just proved, the rows are exact
ii) the first and the fourth vertical maps are isomorphisms by lemma 1
iii) the second vertical map is so by the proof of theorem 1
iv) the first square commutes by lemma 2b
v) the second square commutes by lemma 2c
vi) the third square commutes by lemma 2a and lemma 3a.

The stated isomorphism now follows by application of the five lemma.

Q.E.D.

(To be continued)